

Hyperscaling in the Domany-Kinzel cellular automaton

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An apparent violation of hyperscaling at the end point of the critical line in the Domany-Kinzel stochastic cellular automaton finds an elementary resolution upon noting that the order parameter is discontinuous at this point. We derive a hyperscaling relation for such transitions and discuss applications to related examples.

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Domany and Kinzel's stochastic cellular automaton (DKCA) continues to attract interest as a particle system affording a test of universality in nonequilibrium critical phenomena [1, 2]. In the phase diagram of the DKCA, a line of critical points separates the active phase from the absorbing vacuum phase. There is good numerical evidence [2-4] that the critical behavior along this line is that of directed percolation (DP), except at one terminal point (called *compact directed percolation* or CDP), where the asymptotic behavior is known exactly. Essam showed that the critical exponents governing the moments of the cluster-size distribution in CDP violate the usual hyperscaling relation, and derived the appropriate generalization [5]. In this Brief Report we point out that the hyperscaling relation among the exponents describing spreading also requires modification. This leads us to formulate a hyperscaling relation for *discontinuous* transitions into an absorbing state.

In the one-dimensional DKCA, the state $\sigma_i(t+1)$ ($=0$ or 1) of site i at time $t+1$ depends upon $T_i \equiv \sigma_i(t) + \sigma_{i-1}(t)$. If we let $\text{Pr}[\sigma_i(t+1) = 1] = h(T_i)$, then $h(0) = 0$, $h(1) = p_1$, and $h(2) = p_2$. (All sites are updated simultaneously in this discrete-time process.) For each $p_2 < 1$ there is a critical value $p_{1,c}$, such that δ_ϕ (the vacuum — all sites in state 0), is the unique steady state for $p_1 < p_{1,c}$, while for $p_1 > p_{1,c}$ there is also an "active" stationary state with a nonzero density of sites in state 1. It is believed that the critical behavior along the line $p_{1,c}(p_2)$ is the same as that of DP, for $p_2 < 1$.

The case $p_2 = 1$ (CDP) is special. On this line, δ_1 , the all-1 state, is absorbing, as well as δ_ϕ . The evolution is determined by the coalescence of random walks marking the boundaries between strings of 0's and 1's; for $p_1 = 1/2$ the walks are *unbiased*. In fact, $(p_1 = 1/2, p_2 = 1)$ is a critical end point: for $p_1 < 1/2$, δ_ϕ is the globally attractive stationary state; and conversely for δ_1 , when $p_1 > 1/2$. (By "globally attractive" we mean that for

$p_1 < 1/2$, initial configurations with an infinite number of 0's evolve to δ_ϕ with probability 1, and that there is a nonzero probability to reach δ_ϕ if there is at least one site in state 0 initially.) Let $p_1 = 1/2 + \Delta$. It is easy to see that CDP is invariant under the transformation which takes Δ to $-\Delta$ and interchanges 0's and 1's.

Consider the evolution at $(1/2, 1)$, starting with the origin in state 1, all other sites zero. The number $n(t)$ of 1's at time t is a simple random walk on \mathbb{Z} starting at $n(0) = 1$, with an absorbing boundary at the origin. From well-known properties of random walks, we have that the survival probability $P(t) = \text{Pr}[n(t') > 0, t' \leq t] \sim t^{-1/2}$, and that the mean-square population of 1's in *surviving* trials is $n_{\text{surv}}(t) = E[n(t)^2 | n(t') > 0, t' \leq t] \sim t$. It follows that the mean number of 1's (averaged over all trials, including those that die out) is asymptotically $O(1)$. Spreading of a critical process from a localized source is customarily described by a set of exponents defined *via* the relations [6]

$$P(t) \propto t^{-\delta}, \quad (1)$$

$$n(t) \propto t^\eta, \quad (2)$$

and

$$R^2(t) \propto t^z, \quad (3)$$

where $n(t)$ denotes the mean population size and $R^2(t)$ the mean-square spread of particles (1's, in our notation) about the origin. Along the critical line in the DKCA, the exponents δ , η , and z take universal values (i.e., those of 1+1-dimensional DP), *except* at $(1/2, 1)$, where evidently $\delta = 1/2$, $\eta = 0$, and $z = 1$.

The spreading exponents are connected by a hyperscaling relation [6]

$$4\delta + 2\eta = dz, \quad (4)$$

where d is the number of spatial dimensions. While the exponents for DP are consistent with this relation, those for the point $(1/2, 1)$ (with $d = 1$) are not. Does this signal a violation of hyperscaling in CDP? Here it is useful

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to note that Eq. (4) is a special case of a more general relation [7]

$$\left(1 + \frac{\beta}{\beta'}\right) \delta + \eta = \frac{dz}{2}, \quad (5)$$

where β is the usual order-parameter exponent, defined, for the DKCA, through $\rho_1 \propto (p_1 - p_{1,c})^\beta$, for $p_1 > p_{1,c}$; ρ_1 is the stationary density of 1's. β' governs the ultimate survival probability (starting from a localized source): $P_\infty \equiv \lim_{t \rightarrow \infty} P(t) \propto (p_1 - p_{1,c})^{\beta'}$. Equation (5) reduces to Eq. (4) when the order parameter and P_∞ are governed by the same exponent, as they are in the contact process and other DP-like models. Models with multiple absorbing configurations [7], and branching annihilating random walks with even parity [8], can have $\beta \neq \beta'$, in which case the exponents violate Eq. (4), but satisfy Eq. (5).

It is known that $\beta' = 1$ in CDP [1, 5, 9, 10]. The order-parameter exponent, β , by contrast, is zero. This is because (1/2,1) marks a *discontinuous* transition, as is readily seen by recalling the symmetry property noted above. Since $\rho_1 = 0$ for $p_1 < 1/2$ (the globally attractive state is δ_ϕ), it follows that $\rho_1 = 1$ for $p_1 > 1/2$. Strictly speaking, β is not defined here. But since $\lim_{\Delta \searrow 0} \Delta^\beta = 1$ for any $\Delta > 0$, it is natural to associate the value $\beta = 0$ with the discontinuous transition. Indeed, the values $\delta = 1/2$, $z = d = 1$, and $\eta = \beta = 0$ yield an identity when inserted in Eq. (5).

In fact, we can eliminate any reference to the ill-defined exponent β by adapting Grassberger and de la Torre's scaling argument to discontinuous transitions. Consider a model with a transition from an absorbing to an active state at $\Delta = 0$, with exponents δ , η , z , and β' defined as above. Suppose, however, that the order parameter ρ is discontinuous, being zero for $\Delta < 0$, and

$$\rho = \rho_0 + f(\Delta), \quad (6)$$

for $\Delta > 0$, where $\rho_0 > 0$, and f is continuous and vanishes at $\Delta = 0$. The scaling hypothesis for spreading from a source postulates the existence of two scaling functions, defined via [6]

$$\rho(x, t) \sim t^{\eta - dz/2} G(x^2/t^z, \Delta t^{1/\nu_{||}}), \quad (7)$$

and

$$P(t) \sim t^{-\delta} \Phi(\Delta t^{1/\nu_{||}}). \quad (8)$$

[Here $\rho(x, t)$ is the local order-parameter density. $\nu_{||}$ governs the divergence of the correlation time τ at the transition: $\tau \sim \Delta^{-\nu_{||}}$.] The existence of the limit P_∞ implies that $\Phi(x) \sim x^{\beta'}$ as $x \rightarrow \infty$, with $\beta' = \delta\nu_{||}$. In a surviving trial, the local density must approach the stationary density ρ as $t \rightarrow \infty$, so $\rho(x, t) \sim \Delta^{\beta'} \rho_0$ for $t \rightarrow \infty$ with fixed x and Δ small but positive. It follows that $G(0, y) \sim y^{\beta'}$ for large y . On the other hand, we must have $G(0, y) \sim y^{-\nu_{||}(\eta - dz/2)}$ for $\lim_{t \rightarrow \infty} \rho(x, t)$ to exist. Comparing these asymptotic behaviors, we find a hyperscaling relation for transitions at which the order parameter is discontinuous,

$$\delta + \eta = \frac{dz}{2}. \quad (9)$$

The interpretation is immediate: simply note that $\delta + \eta$ is the exponent governing the mean population in surviving trials, and that the radius R_t of such a cluster grows $\sim t^{z/2}$. (Since the density is positive, clusters are compact, not fractal as in DP.) Equation (9) is nothing more than the scaling law for the volume of a d -dimensional sphere of radius R_t . It should apply whenever power-law growth produces *compact colonies*. (By ‘‘colony’’ we mean the set of particles, or 1's — in general, active sites — at time t , descended from a single active site at $t = 0$. ‘‘Compact’’ means that the density in surviving colonies remains finite as $t \rightarrow \infty$.)

A closely related example is the *voter model* [9, 11]. In this continuous-time Markov process sites of \mathbb{Z}^d are either in state 0 or state 1. If site i is in state 0, it changes to 1 at rate λr_1 , where r_1 is the number of nearest neighbors in state 1. (Note that only sites at cluster boundaries can change state.) Similarly, sites in state 1 change to 0 at rate r_0 . (The case with $\lambda \neq 1$ is usually called the *biased voter model*, for obvious reasons.) Clearly, δ_ϕ (δ_1) is the attractive stationary state for $\lambda < 1$ ($\lambda > 1$) so ρ_1 jumps from 0 to 1 at $\lambda = 1$. The one-dimensional voter model is essentially a continuous-time version of CDP with $p_1 = \lambda/(1 + \lambda)$. Thus we expect all results for critical exponents in the DKCA to apply as well to the voter model in one dimension.

We can analyze spreading in the (unbiased) voter model in $d \geq 2$ as follows. Let n_s be the number of 1's after s changes in the configuration; $n_0 = 1$. n_s is again a simple random walk on \mathbb{Z} , with an absorbing boundary at the origin, so we know that the survival probability $\sim s^{-1/2}$, the mean-square displacement over surviving walks $\sim s$, and the mean displacement over all walks is $O(1)$. The latter implies $\eta = 0$. The number of steps per unit time is proportional to the boundary (number of 0-1 nearest neighbor pairs), b_s :

$$\frac{dt}{ds} \sim \frac{1}{b_s}. \quad (10)$$

In one dimension, $b_s = 2$, so $t \sim s$. For $d \geq 2$ the boundary depends on the shape and internal structure of a colony; we assume $b_s \sim n_s^\gamma$ as $s \rightarrow \infty$. Then since $n_s \sim s^{1/2}$, we have $t(s) \sim s^{1-\gamma/2}$, $n_{surv}(t) \sim t^{1/(2-\gamma)}$, and $P(t) \sim t^{-1/(2-\gamma)}$, so that $\delta = 1/(2-\gamma)$. Since $b \leq qn$ on a lattice with coordination number q , $\gamma \leq 1$, implying that $\frac{1}{2} \leq \delta \leq 1$.

We expect asymptotic properties to be captured by a continuum description, which for the voter model takes the form of a very simple stochastic partial differential equation,

$$\frac{\partial \rho(x, t)}{\partial t} = \nabla^2 \rho(x, t) + \eta(x, t), \quad (11)$$

where $\rho(x, t)$ is a coarse-grained density of 1's and the (Gaussian) noise has autocorrelation $\langle \eta(x, t) \eta(x', t') \rangle = \delta(x - x') \delta(t - t') \rho(x, t) [1 - \rho(x, t)]$. (There are no fluctuations when the density is pinned at one of the absorbing values.) In the mean-field approximation (neglecting the noise) the population spreads diffusively and $z = 1$. But

we know that $z = 1$ even for $d = 1$, suggesting that the mean-field exponent is correct for *all* d .

Simulations of the voter model in two and three dimensions support this conjecture. In two dimensions, a study of 2×10^6 independent realizations (all starting with a single 1), up to a maximum time of 10^4 , yield $\delta = 0.95(3)$, $\eta = -0.01(1)$, and $z = 0.99(2)$ (figures in parentheses indicate statistical uncertainties). In three dimensions (10^6 realizations up to $t = 900$), we obtain $\delta \simeq 0.96(2)$, $\eta = 0.00(1)$, and $z = 1.00(1)$. (In both two and three dimensions the colony mean-square radius of gyration grows linearly with time.) Moreover, the boundary exponent γ appears to be unity in each case. In two dimensions, γ shows considerable time dependence, increasing monotonically from 0.82 for $t = 1000$ to 0.93 for $t = 20\,000$, the longest study attempted. In three dimensions our study of colonies surviving to $t = 900$ yields

$\gamma = 0.975(4)$. The simplest conclusion is that $\gamma = \delta = 1$ for $d \geq 2$. Then the hyperscaling relation, Eq. (9), is violated for $d > 2$, marking $d = 2$ as a kind of upper critical dimension for spreading in the voter model (consistent as well with the slow approach to asymptotic scaling found in the two-dimensional simulations). The density within a colony $\sim n_{surv}(t)/R_t^d \sim t^{1-d/2}$ when $\eta = 0$ and $\delta = z = 1$, so for $d > 2$ colonies are not compact, and Eq. (9) is not expected to hold.

In summary, we have shown how hyperscaling applies to critical spreading at discontinuous transitions, and discussed applications to compact directed percolation and the closely related voter model.

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